# Jacobian Elliptic Function Solution of Characteristics of Rectangular Groove Wave-guide with Rounded Internal Corners

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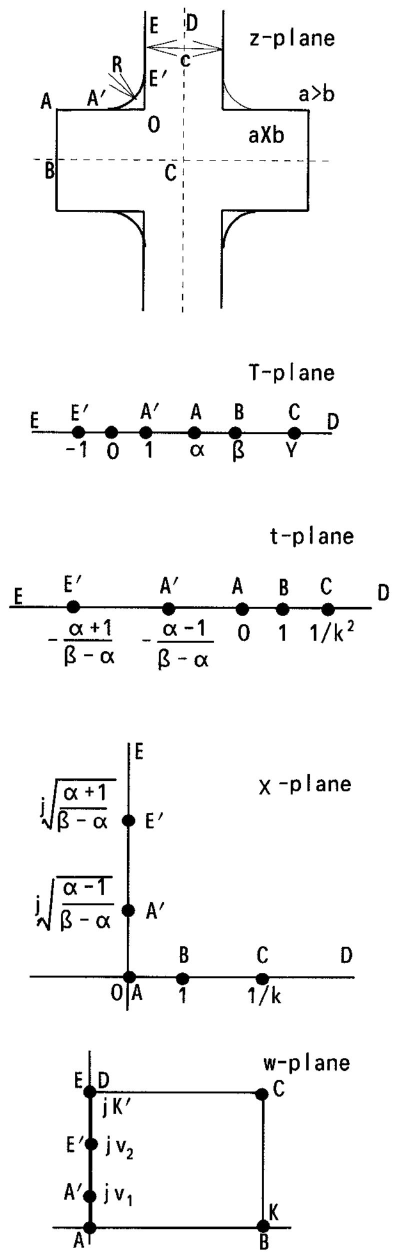
## ABSTRACT

To enhance the power carrying capacity and the efficiency of the rectangular groove wave guide, presented is such a guide with rounded internal corners, by means of Jacobian Elliptic Functions to transform the unbounded waveguide region onto the interior of a rectangle in w-plane with sides 2K(k) × 2K'(k).




Rectangular or groove guides have sharp corners in the guide proper so are not low loss and high power carrying wave guides, theoretically. Proposed now is a groove rectangular waveguide with rounded internal angles, as shown in the figure, where R is radius of the circular rounded corner inside the guide. We map one quarter of this rectangular groove guide with rounded internal angles in the z-plane onto the upper half plane of the T-plane and proceed as Smythe dose (W.R.Smythe, Statics and Dynamic Electricity, § 4.28), by the following equation





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$$dz = A_1 \frac{(T+1)^{1/2} + \lambda(T-1)^{1/2}}{\{(T-\alpha)(T-\beta)(T-\gamma)\}^{1/2}} dT$$
$$= KA_1 \frac{(t+m)^{1/2} + \lambda(t+n)^{1/2}}{\{t(1-t)(1-k^2 t)\}^{1/2}} dT \quad (1)$$

by a change of variable

$$t = \frac{T-\alpha}{\beta-\alpha},\ k^2 = \frac{\beta-\alpha}{\gamma-\alpha},\ \frac{m}{n} = \frac{\alpha \pm 1}{\beta-\alpha}.$$

We have to find five constant $A_1$, $\lambda$, $\alpha$, $\beta$ and $\gamma$, from the points $\infty$, $jR$, $-R$, $-\frac{a-c}{2}$, $-\frac{a-c}{2} - j\frac{b}{2}$ and $\frac{c}{2} - j\frac{b}{2}$. First, we let v → ∞, then (1) gives $\frac{A_1(1+\lambda)T^{1/2}}{T^{3/2}} Tjd\theta = dU$, integrating, we find

$$A_1 = j\frac{c}{2\pi(1+\lambda)} \quad (2)$$

where c is the width of the groove. To make use of the other points, we make

$$t = \chi^2,\ \chi = sn(w,k),\ w = u + jv \quad (3)$$

and in the $\chi$ and $w$ planes points on the boundary of the z-plane are as shown, (1) becomes

$$dz = 2A_1 k\left\{\sqrt{m + sn^2(w,k)} + \lambda\sqrt{n + sn^2(w,k)}\right\}dw \quad (4)$$

and the values of $u_1$ and $u_2$ can be found as follows

$$j\sqrt{\frac{\alpha+1}{\beta-\alpha}}sn(u+jv,k) = \frac{sd_1 + jcds_1c_1}{1 - s_1^2 d^2} \quad (5)$$

where $s = sn(v,k)$, $s_1 = sn(u,k')\cdots$, so $s(u,k) = 0$, u=0, and (5) gives

$$\sqrt{\frac{\alpha+1}{\beta-\alpha}} = \frac{c_1 s_1}{1-s_1^2} = \frac{s_1(v_2)}{c_1(v_2)} = tn(v_2),\ v_2 = tn^{(-1)}\left(\sqrt{\frac{\alpha+1}{\beta-\alpha}}, k'\right),$$

and similarly $v_1 = sn^{-1}\left(\sqrt{\frac{\alpha-1}{\beta-1}}, k'\right)$. We can now integrate (5), starting from E' to C as follows

$$-R(1+j) = \int_{jv_2}^{jv_1} \frac{dz}{dw} dw$$
$$= \int_{v_2}^{v_1}\left\{\sqrt{m - \frac{sn^2(v_1,k)}{cn^2(v,h)}} + \sqrt{n - tn^2(v,k)}\right\} j2A_1 kdv$$

$$-\frac{a-c}{2} + R = \int_{v_1}^{jv_2}\left\{\sqrt{m - tn^2 v} + \sqrt{n - tn^2 v}\right\} jA_1 kdv$$

$$-j\frac{b}{2} = \int_0^K \left\{\sqrt{m + sn^2(u,k)} + \sqrt{n + sn^2(u,k)}\right\} 2A_1 kdv$$

$$-\frac{a}{2} = \int_K^{K+jK'} \frac{dz}{dw} dw$$
$$= \int_0^{K'}\left\{\sqrt{m + dn^2(y,k')} + \lambda\sqrt{n + dn^2(y,k')}\right\} j2A_1 kdy$$

$(by\ \ w = K + jy)$

Solving these four equations numerically, we obtain $\alpha$, $\beta$, $\gamma$ and $\lambda$. We study the weighted wave equation of the rectangular groove guide with rounded internal angles

$$\frac{\partial^2 H_z}{\partial u^2} + \frac{\partial^2 H_z}{\partial v^2} + h^2(\gamma^2 + k^2)H_z = 0,\ h^2 = \left|\frac{dz}{dw}\right|^2$$

No exact solution of this equation can be founded, and we have to resort to numerical or approximate solutions. For example, in the zero order approximation we can neglect $sn^2(w,k)$ in (4) so h is a constant and this wave equation gives sinusoidal waves in a rectangular guide of dimensions of 2K(k) × 2K' (k) in the w-plane as shown in the figure.

By transforming the boundary in the $\chi = \sqrt{t}$ plane by the Jacobian elliptic function, we transform the unbounded guide region in the z-plane into the bounded region of the interior of a rectangle in the w-plane, we are then in a better position to solve the weighted wave equation by numerical or approximate method.